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THE CONSTRUCTION OF OPTIMAL MODES FOR EXCITING OSCILLATIONS OF A TWO-LINK PHYSICAL PENDULUM†

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Optimal control problems for a plane two-link pendulum whose links are homogeneous thin rods (material segments) are solved. The control variable is either the angular velocity of rotation of the second link relative to the first (Problem 1) or the angle between the two links (Problem 2). It is assumed that at the starting time the following parameter values are given: the angle of deviation of the first link (assumed to be positive), the angle between the links, and the angular momentum of the system relative to the suspension point (which is assumed to be zero). It is required to minimize (maximize in absolute value) the angle of deviation of the first link from the vertical at the time the angular momentum first becomes zero after the pendulum is set in motion. In Problem 1 a certain restriction is imposed on the absolute value of the control function. The Pontryagin maximum principle is used to show that the optimal control consists of alternating non-singular and singular portions. The behaviour of the optimal control is investigated as a function of the maximum admissible angular velocity of relative rotation of the links. Problem 2 is solved with no restrictions, using methods of the calculus of variations. A boundary-value problem also arises here, which is equivalent to the boundary-value problem of the maximum principle for the case in which the control occurs on the right-hand sides of the differential equations of motion together with its derivatives and is not subject to any restrictions. A numerical algorithm is proposed to solve the boundary-value problem. An example is presented in which an optimal control is constructed for a pendulum with particular parameters. It is shown that in optimal motion the angle between the links varies continuously everywhere, except at the initial and final instants of the control process, where the angle varies by a jump.

One reason why it is difficult to investigate oscillatory mechanical systems with several degrees of freedom effectively is the large number of dimensions of the corresponding system of differential equations of motion. The dimensionality of the system of equations may be reduced if one of the phase coordinates is chosen as a control. When that is done, however, new mechanical effects and new forms of optimal control, not present in the original system, may appear.

In this paper, taking as an example a plane two-link physical pendulum, we present the results of a complex investigation of the initial system of equations of motion and a corresponding system of equations of lower order. The first link of the pendulum may oscillate about a fixed point, while the second oscillates about the first. Only oscillatory motions will be considered. Two optimal control problems will be formulated and solved, both relating to the maximum deviation of the first link from the vertical over one half-cycle of the pendulum's swing. In Problem 1 the control function is the angular velocity of rotation of the second link relative to the first. It is assumed that the control in bounded. The problem has a unique solution. The optimal control function contains alternating portions of non-singular and singular control.

In Problem 2, the control is the angle of rotation of the second link relative to the first. This problem is a limiting case of Problem 1 when the angular velocity of the second link may take absolute values as large as desired. It is shown that the problem is multi-extremal and has an infinite set of solutions. New effects, characteristic of Problem 2 only, are demonstrated. The process of continuous transition from the solution of the first problem to a solution of the second is described.

The algorithms developed to solve the boundary-value problems are extensions of earlier results [4].

The problem of the swinging if a two-link pendulum is related to the problem of optimizing the sportive motions of an athlete on a crossbar. The problem is also of interest in theoretical mechanics. Analysis of the results obtained improves our understanding of the problem of investigating and constructing optimal oscillatory modes of motion in mechanical systems.

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1. OPTIMAL CONTROL OF THE ANGULAR VELOCITY OF THE SECOND LINK OF THE PENDULUM

Mathematical model of the motion. Consider a plane two-link pendulum whose links are homogeneous rods of length l_1 and l_2 and mass m_1 and m_2 (Fig. 1). At the suspension point 0 of the first link and between the links there are single degree-of-freedom joints. The angle of deviation of the first link from the vertical is denoted by φ_1 and that of the second link from the continuation of the first by φ_2. The angles are measured counterclockwise.

The angular momentum K of the two-link pendulum about the point O may be written in the form

$$K = A(\varphi_2)\dot{\varphi}_1 + B(\varphi_2)\dot{\varphi}_2 \tag{1.1}$$

where

$$A(\varphi_2) = a_0 + a_1\cos\varphi_2, \quad B(\varphi_2) = b_0 + b_1\cos\varphi_2 \tag{1.2}$$

$$a_0 = \frac{1}{3}m_1 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1^2, \quad a_1 = m_2 l_1 l_2, \quad b_0 = \frac{1}{3}m_2 l_2^2, \quad b_1 = \frac{1}{2}m_2 l_1 l_2$$

As control function we choose the angular velocity of rotation of the second link of the pendulum relative to the first: $u = \varphi_2(t)$. The control is assumed to be bounded

$$|u| < c \tag{1.3}$$

The equations of motion in a time interval $0 \leqslant t \leqslant T$, in terms of the variables $(K, \varphi_1, \varphi_2)$, are

$$\dot{K} = L(\varphi_1, \varphi_2), \quad \dot{\varphi}_1 = \frac{K}{A(\varphi_2)} - C(\varphi_2)u, \quad \dot{\varphi}_2 = u \tag{1.4}$$

where

$$k_1 = -\frac{1}{2}m_1 l_1 g - m_2 l_1 g, \quad k_2 = -\frac{1}{2}m_2 l_2 g$$

$$L(\varphi_1, \varphi_2) = k_1\sin\varphi_1 + k_2\sin(\varphi_1 + \varphi_2), \quad C(\varphi_2) = \frac{B(\varphi_2)}{A(\varphi_2)}$$

The initial conditions are assumed to be given:

$$K(0) = 0, \quad \varphi_1(0) = \varphi_{10} > 0, \quad \varphi_2(0) = \varphi_{20} \tag{1.5}$$

Formulation of problem 1. For the equation system (1.4) with initial conditions (1.5), it is required to construct a control $u(t)$ which satisfies condition (1.3) and, at the time $t = T$ when the angular momentum first vanishes for $t > 0$:

$$F_1 = K(T) = 0 \tag{1.6}$$

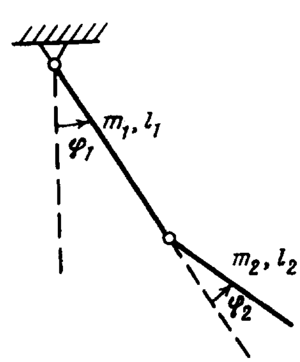

Fig. 1.

guarantees that the first link will deviate from the vertical by the maximum amount:

$$F_0 = -\varphi_1(T) \to \max \tag{1.7}$$

It is assumed that the pendulum's motion in its first swinging half-cycle is oscillatory, and at some $t = T > 0$ the angular momentum vanishes at an angle $\varphi_1(T) \in (-\pi, 0)$.

Necessary conditions for an extremum. The necessary conditions for the existence of an extremum in problem (1.1)–(1.7), in the form of the Pontryagin maximum principle [5], are as follows ($\{\lambda_1(t), \lambda_2(t), \lambda_3(t)\}$ is the vector of conjugate variables):

$$\dot{\lambda}_1 = -\frac{\lambda_2}{A(\varphi_2)}, \quad \dot{\lambda}_2 = -L(\varphi_1, \varphi_2)\lambda_1 \tag{1.8}$$

$$\dot{\lambda}_3 = -k_2 \cos(\varphi_1 + \varphi_2)\lambda_1 - \frac{a_1 K \sin\varphi_2}{[A(\varphi_2)]^2}\lambda_2 + u\frac{k_{01}\sin\varphi_2}{[A(\varphi_2)]^2}\lambda_2 \quad (k_{01} = a_1 b_0 - a_0 b_1)$$

$$\lambda_2(T) = -1, \quad \lambda_3(T) = 0, \quad H = H_0 + H_1 u \to \max, \quad H(T) = 0 \tag{1.9}$$

where

$$H_0 = \lambda_1 L(\varphi_1, \varphi_2) + \lambda_2 \frac{K}{A(\varphi_2)}, \quad H_1 = \lambda_3 - \lambda_2 C(\varphi_2) \tag{1.10}$$

Method of investigating the problem and results of computations. The problem is solved in two stages. In the first stage, the structure of the optimal control law is determined by using a modified method of successive linearization and methods for qualitative analysis of the equations of motion and optimality conditions [4]. It is established that the optimal control function, when $c > 2\pi\ \mathrm{s}^{-1}$, includes three portions of the motion with boundary control $u(t) = -c$ and two portions of motion with internal (singular) control (Fig. 2a).

In the second stage, a numerical solution of the boundary-value problem of the maximum principle is constructed using the following algorithm.

1. Fix a time t_1 and, in the time interval $0 \leqslant t \leqslant t_1$, integrate the basic system of equations (1.4) with control $u(t) = -c$.

2. At $t = t_1$, define $\lambda_2(t_1) = -1$. Using the conditions $H_1(t_1) = 0$ (see (1.10)) and $\dot{H}_1(t_1) = 0$, find $\lambda_3(t_1)$ and $\lambda_1(t_1)$, where

$$\dot{H}_1 = \lambda_1 C(\varphi_2) M(\varphi_1, \varphi_2) - \lambda_1 k_2 \cos(\varphi_1 + \varphi_2) - \lambda_2 \frac{a_1 K \sin\varphi_2}{[A(\varphi_2)]^2} \tag{1.11}$$

$$M(\varphi_1, \varphi_2) = k_1 \cos\varphi_1 + k_2 \cos(\varphi_1 + \varphi_2)$$

3. Vary t_1 and successively implement steps 1° and 2° until the condition $H_0(t_1) = 0$ (see (1.10)) is met.

4. Fix a time t_2, and, in the interval $t_1 \leqslant t \leqslant t_2$, integrate the basic system (1.4) together with the conjugate system (1.8) with a singular control [6] computed from the condition

$$\ddot{H}_1 = D_2 u + D_1 = 0 \tag{1.12}$$

where

$$D_1 = -k_1 C(\varphi_2)\left(\frac{\lambda_2 \cos\varphi_1}{A(\varphi_2)} + \frac{K\lambda_1 \sin\varphi_1}{A(\varphi_2)}\right) - \frac{a_1 \lambda_2 \sin\varphi_2}{[A(\varphi_2)]^2}(k_1 \sin\varphi_1 + k_2 \sin(\varphi_1 + \varphi_2)) +$$

$$+ \frac{a_1 K \lambda_1 \sin\varphi_2}{[A(\varphi_2)]^2} M(\varphi_1, \varphi_2) - k_2\left(\frac{\lambda_2 \cos(\varphi_1 + \varphi_2)}{A(\varphi_2)} + \frac{K\lambda_1 \sin(\varphi_1 + \varphi_2)}{A(\varphi_2)}\right)(C(\varphi_2) - 1)$$

$$D_2 = \frac{\lambda_1 \sin\varphi_2}{[A(\varphi_2)]^2} k_{01} M(\varphi_1, \varphi_2) + \lambda_1 k_1 [C(\varphi_2)]^2 \sin\varphi_1 +$$

$$+ \lambda_1 k_2 \sin(\varphi_1 + \varphi_2) C(\varphi_2)(C(\varphi_2) - 1)^2 - \frac{a_1 K \lambda_2}{[A(\varphi_2)]^3}(a_1 + a_1 \sin^2\varphi_2 + a_0 \cos\varphi_2)$$

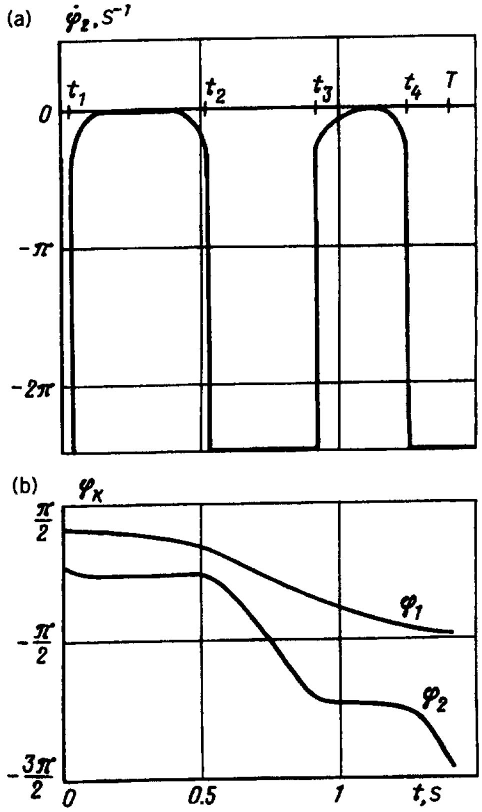

Fig. 2.

5. Fix a time, t_3 and, in the interval $t_2 \leqslant t \leqslant t_3$, integrate the basic and conjugate systems with control $u(t) = -c$.

6. Vary the times t_2, t_3 and successively implement steps 4° and 5° until the conditions $H_1(t_3) = 0$, $H_1(t_3) = 0$ are met.

7. Fix a time t_4 and, in interval $t_3 \leqslant t \leqslant t_4$, integrate the basic and conjugate systems with a singular control computed from condition (1.12).

8. Fix a time T and, in the interval $t_4 \leqslant t \leqslant T$, integrate the basic and conjugate systems with the control $u(t) = -c$.

9. Vary the times t_4, T and successively implement steps 7° and 8° until condition (1.6) and the second condition of (1.9) are met.

10. Integrate the conjugate system in reverse time in the interval $0 \leqslant t \leqslant t_1$, with the boundary conditions computed at step 2 and the control $u(t) = -c$. Construct the vector function of conjugate variables $\{\lambda_1(t), \lambda_2(t), \lambda_3(t)\}$ over the entire portion of the motion, $0 \leqslant t \leqslant T$.

11. For the interval $0 \leqslant t \leqslant T$, verify that the last two conditions of (1.9) and Kelly's condition $D_2 > 0$ hold for the singular controls.

Note that the last condition of (1.9) will be satisfied by virtue of step 2°, but Kelly's condition and the multimate condition of (1.9) will hold only provided that the structure of the optimal control law was correctly established at the first stage of the investigation by the direct method of [4]. The first condition of (1.9) may be ensured by suitable normalization of the vector $\{\lambda_1(t), \lambda_2(t), \lambda_3(t)\}$.

The results of the computations will be illustrated for a model with the following parameters: $l_1 = 2$ m, $l_2 = 0.4$ m, $m_1 = 10$ kg, $m_2 = 4$ kg; $\varphi_{10} = \pi/3$ and $\varphi_{20} = 0$. The number c is varied. With these parameters, the function $A(\varphi_2)$(1.2) does not vanish for any value of the angle φ_2. Consequently, the right-hand side of the second equation of (1.4) is a continuously differentiable function of φ_2.

At $c = 2.5\ \pi\ s^{-1}$, the maximum deviation of the first link is $\varphi_1(T) = -1.318$. The optimal control $u(t) = \dot{\varphi}_2(t)$ is shown in Fig. 2(a) and the corresponding optimal $\varphi_1(t)$ and $\varphi_2(t)$ are shown in Fig. 2(b). As the parameter c increases, the duration of each of the portions with control $u(t) = -c$ decreases while that of the two portions of singular control increases. If the parameter c increases without limit, the control function $u(t)$ approaches infinity in absolute value at the points $t = 0$ and $t = T$ and is fairly large, but finite, approximately at $t = T/2$. Throughout the other portions of the motion the control varies smoothly and is close to zero.

Note that if restriction (1.3) on the angular velocity of motion of the second link is dropped, problem (1.1)–(1.7) becomes equivalent to the problem considered below—the optimal control of the angle of rotation of the second link.

2. OPTIMAL CONTROL OF THE ANGLE OF ROTATION OF THE SECOND LINK OF THE PENDULUM

Mathematical model of the motion of the pendulum, assuming that the angle of rotation of the second link changes abruptly. Let us assume that the motion of the pendulum, when the position of the second link changes abruptly, obeys the law of conservation of angular momentum. We multiply the left- and right-hand sides of the expression for the angular momentum (1.1) by dt and write the result as $Kdt = A(\varphi_2)d\varphi_1 + B(\varphi_2)d\varphi_2$. After determining $d\varphi_1$ from this expression, integrating over the time interval from t to $t + \Delta t$ and letting $\Delta t \to 0$, we obtain

$$\Delta\varphi_1 = -k_3\Delta\varphi_2 - k_4\left[\operatorname{arctg}\left(\frac{1}{k_5}\operatorname{tg}\frac{\varphi_2+\Delta\varphi_2}{2}\right) - \operatorname{arctg}\left(\frac{1}{k_5}\operatorname{tg}\frac{\varphi_2}{2}\right)\right] \tag{2.1}$$

$$k_3 = \frac{b_1}{a_1}, \quad k_4 = \frac{2k_{01}}{a_1\sqrt{a_0^2 - a_1^2}}, \quad k_5 = \sqrt{\frac{a_0 + a_1}{a_0 - a_1}}$$

Figure 3, for the interval $-2\pi \leqslant \Delta\varphi_2 \leqslant 0$, plots $\Delta\varphi_1$ as a function of $\Delta\varphi_2$ for various values of φ_2. The function $\Delta\varphi_1(\Delta\varphi_2)$ (2.1) takes its extremum values at the points where $B(\varphi_2 + \Delta\varphi_2) = 0$. Note that in the model under consideration $B(\varphi_2 + \Delta\varphi_2) = 0$ for $\varphi_2 + \Delta\varphi_2 = \pm 1{,}738 + 2\pi n, n \in Z$. For example, at the starting time, when $\varphi_2 = 0$ function (2.1) reaches a local maximum $\Delta\varphi_1 = 0.078$ when the second link turns abruptly through an angle $\Delta\varphi_2 = -1.738$. The next local maximum $\Delta\varphi_1 = 0.121$ is obtained at $\Delta\varphi_2 = -8.021$. Note that abrupt rotation of the second link through an angle $\Delta\varphi_2 = -2\pi$ implies a jump in the angle φ_1 through $\Delta\varphi_1 = 0.043$. Analogous computations yield the local maxima of the function $\Delta\varphi_1(\Delta\varphi_2)$ at time $t = T$.

Thus, by "fast rotation of the second link" the deviation of the first link at any time $t \in [0, T]$ (in particular, at the initial and final times) may instantaneously be made as large in absolute value as desired. Consequently, the problem of global maximization of the deviation of the first link by controlling the angle of rotation of the second link over a single swinging half-cycle is degenerate (multi-extremal). However, for the practical construction of almost-optimal controls, we are also interested in the pendulum motions over the time interval $0 \leqslant t \leqslant T$ in which the deviation of the first link reaches a local maximum at $t = T$. In what follows, we shall call these motions local extremals or simply extremals.

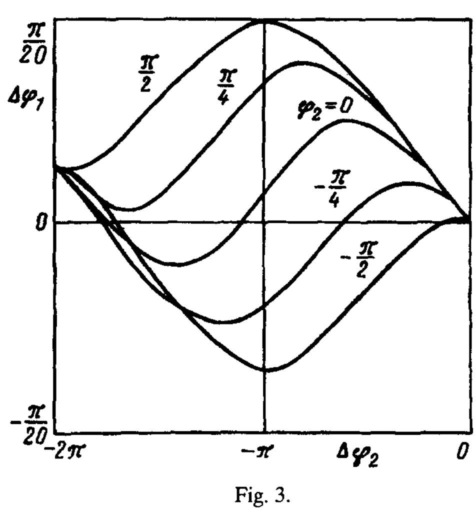

Fig. 3.

Naturally, they must satisfy appropriate necessary conditions of optimality.

Note that the jump components of the extremals at $t = 0$, which ensure that the first link will experience locally maximum deviations when the control extends over the whole interval of motion $0 \leqslant t \leqslant T$, do not coincide with the local maxima of the function $\Delta\varphi_1(\Delta\varphi_2)$ (2.1) at $t = 0$.

Formulation of Problem 2. For the system of differential equations consisting of the first two equations of (1.4), with initial conditions (1.5), it is required to construct a control $\varphi_2(t)$ which, at the time $t = T$ when the angular momentum first vanishes for $t > 0$ (1.6), imparts to the first link of the pendulum the maximum deviation from the vertical (1.7).

It is assumed that, as in problem (1.1)–(1.7), such a time $t = T$ exists.

Note that in Problem 2 the right-hand sides of the differential equations of motion involve both the control function $\varphi_2(t)$ itself and its derivative $\varphi_2(t)$[7].

Necessary conditions for an extremum. We will write the functional of the problem as

$$J = \int_0^T \lambda_1(\dot{K} - k_1 \sin\varphi_1 - k_2 \sin(\varphi_1 + \varphi_2))dt +$$

$$+\int_0^T \lambda_2\left(\dot{\varphi}_1 - \frac{K}{A(\varphi_2)} + C(\varphi_2)\dot{\varphi}_2\right)dt - g_0\varphi_1(T) + g_1 K(T) \tag{2.2}$$

where g_0 and g_1 are as yet undetermined constants.

The variation of the functional may be expressed in the form

$$\delta J = \lambda_1(T)\delta K(T) - \lambda_1(0)\delta k(0) - \int_0^T \dot{\lambda}_1 \delta K dt + \lambda_2(T)\delta\varphi_1(T) - \lambda_2(0)\delta\varphi_1(0) - \int_0^T \dot{\lambda}_2 \delta\varphi_1 dt +$$

$$+\int_0^T \lambda_1(-M(\varphi_1,\varphi_2)\delta\varphi_1 - k_2\cos(\varphi_1+\varphi_2)\delta\varphi_2)dt + \int_0^T \lambda_2\left(-\frac{\delta K}{A(\varphi_2)} - \frac{Ka_1\sin\varphi_2\delta\varphi_2}{[A(\varphi_2)]^2}\right)dt +$$

$$+\lambda_2(T)C(\varphi_2(T))\delta\varphi_2(T) - \lambda_2(0)C(\varphi_2(0))\delta\varphi_2(0) - \int_0^T \dot{\lambda}_2 C(\varphi_2)\delta\varphi_2 dt - g_0\delta\varphi_1(T) + g_1\delta K(T) +$$

$$+\lambda_1(T)(\dot{K}(T) - k_1\sin\varphi_1(T) - k_2\sin(\varphi_1(T) + \varphi_2(T)))\delta T +$$

$$+\lambda_2(T)\left(\dot{\varphi}_1(T) - \frac{K(T)}{A(\varphi_2(T))} + C(\varphi_2(T))\dot{\varphi}_2(T)\right)\delta T - g_0\dot{\varphi}_1(T)\delta T + g_1\dot{K}(T)\delta T \tag{2.3}$$

Necessary conditions for an extremum will be derived from the condition $\delta I = 0$. In the process, we will compare them to the corresponding necessary conditions for an extremum in problems (1.1)–(1.7).

Equating the coefficients of δK and $\delta\varphi_1$ in the integral terms of (2.3) to zero, we find the conjugate system of equations. It is identical with the first two equations of the conjugate system (1.8).

Equating the coefficient of δT to zero in the terminal terms of (2.3), we have

$$\dot{\varphi}_1(T)(\lambda_2(T) - g_0) + \dot{K}(T)(\lambda_1(T) + g_1) = 0$$

Since in the general case $\varphi_1(T) \neq 0$ and $K(T) \neq 0$, this implies the following boundary conditions for the conjugate variables

$$\lambda_1(T) = -g_1, \qquad \lambda_2(T) = g_0 \tag{2.4}$$

Equating the coefficient of $\delta\varphi_2$ in the integral terms of (2.3) to zero, we obtain

$$-\dot{\lambda}_2 C(\varphi_2) - k_2\lambda_1\cos(\varphi_1+\varphi_2) - \frac{a_1 K\sin\varphi_2}{[A(\varphi_2)]^2}\lambda_2 = 0$$

After substituting the right-hand side of the second equation of (1.8) for $\dot{\lambda}_2$, we obtain an equation for the control $\varphi_2(t)$ which is identical with the condition $H_1 = 0$ (1.11). Thus, in the optimal control of the angle φ_2 the switching function H_1 must vanish over the whole interval of motion $0 < t < T$, that is, the optimal control function must be singular [6] for problem (1.1)–(1.7).

The transversality conditions at $t = 0$ are

$$\lambda_1(0)\delta K(0) = 0, \quad \lambda_2(0)\delta\varphi_1(0) = 0$$

These conditions will hold due to the first two conditions of (1.5).

The transversality conditions at $t = T$ are

$$(\lambda_1(T) + g_1)\delta K(T) = 0 \qquad (\lambda_2(T) - g_0)\delta\varphi_1(T) = 0$$

They are satisfied by virtue of (2.4).

We now consider the terms

$$\lambda_2(T)C(\varphi_2(T))\delta\varphi_2(T) - \lambda_2(0)C(\varphi_2(0))\delta\varphi_2(0)$$

It follows from the third condition of (1.5) that $\delta\varphi_2(0) = 0$. Consequently, $C(\varphi_2(0))$ does not have to vanish in the general case.

The value of $\varphi_2(T)$ is not fixed, and therefore $\delta\varphi_2(T) \neq 0$. By (2.4), $\lambda_2(T) \neq 0$. Consequently, at the final point it must be true that $C(\varphi_2(T)) = 0$ or

$$B(\varphi_2(T)) = 0 \tag{2.5}$$

Note that this condition, together with (1.1) and (1.6), implies the equality $\varphi_1(T) = 0$.

Since the time of motion T is not fixed and as a result $\delta T \neq 0$, equating the coefficient of δT in the integral terms of (2.3) to zero, we obtain a condition for transversality with respect to time, which, in view of (2.5), is identical with the fourth condition of (1.9).

Thus, the necessary optimum conditions (2.4) and (2.5), together with the previously mentioned necessary conditions for an extremum of Problem 1, determine the boundary-value problem of the maximum principle for Problem 2.

An algorithm for constructing extremals of the problem. Computational results. Based on an analysis of the results of solving problem (1.1)–(1.7), we may assume that the optimal motions of the system consist of jumps at times $t = 0$ and $t = T$ (not uniquely defined) and a portion of continuous control over the interval $0 + 0 < t < T - 0$ (uniquely defined). In practice the extremals with bounded jumps at $t = 0$ and $t = T$ are of particular interest.

Extremals solving the boundary-value problem were constructed in two stages. At the first stage an approximate estimate was found for the parameters $\lambda_2(0)$ and T. At the second stage a two-parametric boundary-value problem with parameters $\lambda_2(0)$ and T was solved and an extremal satisfying the necessary conditions for an extremum was constructed.

We will describe the algorithm used at the first stage to solve the boundary-value problem.

1.1. At time $t = 0$, vary the angle φ_2 by a jump to such a value that, taking (2.1) into account, the condition $H_1 = 0$ (1.11) holds. (The jump of least absolute value is $\Delta\varphi_2 = 0\ -0.191$. In that case the angle φ_1 is instantaneously increased to $\Delta\varphi_1 = 0.013$.)

1.2. Fix λ_1 (0) and some value of the parameter $\lambda_2(0)$.

1.3. Integrate the first two equations of (1.4) and the first two equations of (1.8), with a control $u(t)$, computed from (1.12). Condition (1.6) will be used to determine an approximation for the final time of the motion $t = T$.

1.4. Successively implementing steps 1.2° and 1.3°, construct $\varphi_1(T)$ as a function of the single parameter $\lambda_2(0)$.

1.5. Varying $\lambda_2(0)$ and successively implementing steps 1.2–1.4, find max $\varphi_1(T)$.

This algorithm yields estimates for the parameters $\lambda_2(0)$ and T, which will be used as first approximations in the second stage of the solution of the boundary-value problem.

We now describe the algorithm used in the second stage of the solution of the boundary-value problem.

2.1. Implement step 1.1 of the previously described algorithm.

2.2. Fix $\lambda_1(0) = 1$ and the first approximation of the parameters $\lambda_2(0)$ and T.

2.3. Integrate the first two equations of (1.4) and the first two equations of (1.8) with control $u(t)$ computed from (1.12).

2.4. Vary the values of the parameters $\lambda_2(0)$ and T and successively implement steps 1.2–2.3 until condition (1.6) and the third condition of (1.9) are met.

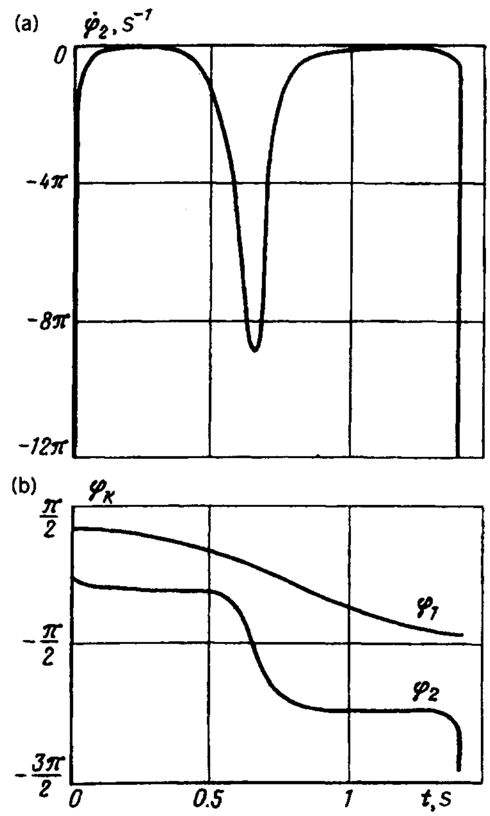

Fig. 4.

2.5. At time $t = T$ change the angle φ_2 abruptly to such a value that the necessary condition (2.5) holds. (The jump of least absolute value is $\Delta\varphi_2 = -1.404$. In that case the angle φ_1 is instantaneously increased in absolute value by $|\Delta\varphi_1| = 0.055$.)

The iterative process just described for constructing extremals converges in 5–10 iterations. Incidentally, the boundary-value problem may also be solved without using the first-stage algorithm, but then the overall computation time is increased. For example, if one takes the initial approximation for $\lambda_2(0)$ and T to be the corresponding quantities from the solution of problem (1.1)–(1.7) with $c = 2.5\ \pi\ s^{-1}$, the system of non-linear equations in the second stage may be solved by Newton's method with a sufficiently small correcting step in 30–35 iterations.

Computations showed that the locally maximum deviation of the first link over one swinging half-cycle of the pendulum, with the minimum possible jumps at times $t = 0$ and $t = T$, amounts to $\varphi_1(T) = -1.375$. Thus, if one drops condition (1.3) in problem (1.1)–(1.7), the maximum deviation of the first link over one swinging half-cycle may increase by $\Delta\varphi_1 = 0.057$ (using "minimum possible" jumps at $t = 0$ and $t = T$). The derivative of the optimal control function $\varphi_2(t)$, as well as the optimal control function $\varphi_2(t)$ and the corresponding optimal function $\varphi_1(t)$, are shown in Fig. 4.

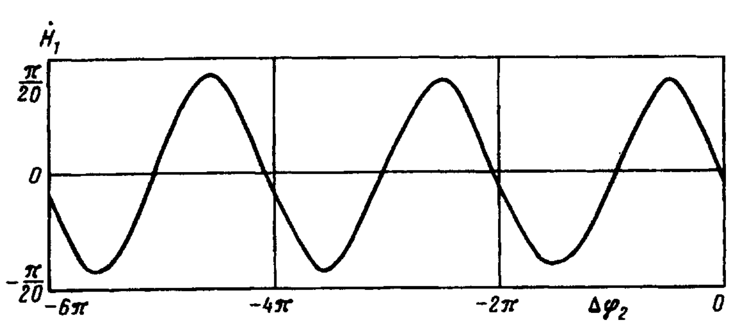

Fig. 5.

We will now describe the special features of the structure of the extremals in this problem. Each extremal consists of three characteristic portions: $0 \leqslant t \leqslant 0+0$, $0+0 \leqslant t \leqslant T-0$, $T-0 \leqslant t \leqslant T$. In the first and third portions the angles φ_2 and φ_1 change abruptly, while in the second portion they are continuous. At $t = 0$ the jump in φ_2 is such that, taking (2.1) into account, the condition $H_1 = 0$ (1.11) holds at $t = 0 + 0$. The function $H_1 = H_1(\Delta\varphi_2)$ (1.11) at $t = 0 + 0$ is shown in Fig. 5. The jump size is clearly not uniquely defined. Locally optimal jumps at $t = 0$ and the corresponding local extremals of the deviation $\varphi_1(T-0)$ for various initial jumps are listed below (the optimal control over the interval $0 + 0 \leqslant t \leqslant T-0$ is uniquely defined if one uses (1.4), (1.8) and (1.12)).

$\Delta\varphi_2$	–0.191	–6.456	–12.721
$\Delta\varphi_1$	0.013	0.054	0.095
$\varphi_1(T-0)$	–1.319	–1.364	–1.474

Obviously, the local maximum points for $\Delta\varphi_2$ differ from one another approximately by an angle of 2π. The slight violation of periodicity is due to the fact that at $t = 0$ the angle φ_1 varies abruptly according to (2.1).

Note that not all the zeros of the function shown in Fig. 5 correspond to extremals of the problem. The only locally optimal jumps are those for which the moment of inertia of the two-link pendulum instantaneously comes sufficiently near the maximum possible, that is, $\Delta\varphi_2 = -2\pi n$, $n \in \mathrm{N}$. In addition, note that these locally optimal jumps do not coincide with the local maxima of the function $\Delta\varphi_1(\Delta\varphi_2)$ shown in Fig. 3.

If the angle φ_2 is abruptly changed at $t = 0$ by a sufficiently large amount, it may turn out that the angle φ_1 jumps to a value of the order of π. In that case this problem—the optimal control of the motion of a two-link pendulum over one swinging half-cycle—becomes meaningless. the pendulum begins to perform rotating motions, which are not being considered here.

At $t = T-0$ the angle φ_2 jumps to value such that the necessary condition (2.5) for an extremum is satisfied at $t = T$. The size of the locally optimal jump in φ_2 is not uniquely defined. The following table lists locally optimal jumps at $t = T-0$ and the corresponding locally optimal deviations of the first link over the entire swinging half-cycle

$\Delta\varphi_2$	–1.404	–7.687	–13.970
$\Delta\varphi_1$	–0.055	–0.099	–0.140
$\varphi_1(T)$	–1.375	–1.419	–1.459

These results correspond to a jump $\Delta\varphi_2 = -0.191$ at time $t = 0$. Note that the points of the locally maximum jumps $\Delta\varphi_2$ at $t = T-0$ differ from one another by an angle of 2π.

Note also that not all roots $\Delta\varphi_2$ of the equation $B(\varphi_2 + \Delta\varphi_2) = 0$ at a time $t = T$ ensure maximum deviation of the first link. The only locally maximum roots are those for which the function (2.1) reaches a local minimum. At $t = T$ the local minima of the function (2.1) (Fig. 3) coincide with the locally optimal jumps in φ_2.

Thus, if no restrictions are imposed on the size of the jump in the angle φ_2 at time $t = T-0$, the deviation of the first link at $t = T$ may be made as large as desired without violating the necessary conditions for an extremum.

The main qualitative result is that, if no restrictions are imposed on the instantaneous jumps at $t = 0$ and $t = T$, one can construct an infinite set of local extrema which solve the problem and satisfy the necessary conditions for an extremum. If a restriction is imposed on the size of the jump, only a finite number of extremals remain. If sufficiently strong restrictions are imposed on the size of the instantaneous jumps (e.g. $|\Delta\varphi_2| < \pi/2$ at $t = 0$ and $t = T$), then the optimal control problem under consideration will have a unique solution, which is the limit obtained from problem (1.1)–(1.7) when $c = \infty$ (see (1.3)). But if jumps in φ_2 are entirely forbidden (this is equivalent to introducing restriction (1.3) on the angular velocity of the second link), the problem becomes equivalent to problem (1.1)–(1.7).

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